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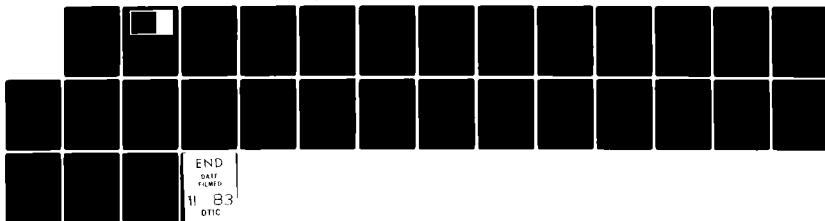
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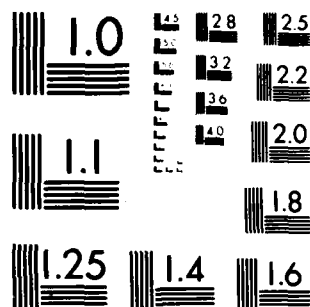
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MRC Technical Summary Report # 2575

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FLUID

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September 1983

(Received July 27, 1983)

SEP 1983

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ABSTRACT

This paper is concerned about the mathematical aspects of the two model equations which describe nonlinear wave motions in a rotating fluid. We establish the local existence of solutions and show that singularities occur in a finite time under certain hypotheses. We also show that these equations admit nonconstant travelling wave solutions.

AMS(MOS) Subject Classification: 35Q20, 76B15, 35B10, 35B32, 35B40

Key Words: nonlinear, dispersive waves in a rotating fluid; local existence of solution; formation of singularities; periodic travelling wave solutions

Work Unit Number 1 - Applied Analysis

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041. This material is based upon work supported by the National Science Foundation under Grant No. MCS-7927062, Mod. 2

SIGNIFICANCE AND EXPLANATION

We study the model equations which describe axially symmetric long wave motions arising in a rotating fluid; see (0-1), (0-2) in the Introduction. We discuss qualitative behavior of these model equations and compare them with the K-dV equation and Burgers' equation (without dissipation). The dispersion mechanism of our model equation is weaker than that of the K-dV equation (see BBM [2]); we show that the local solution (in a reasonably smooth function space) cannot be extended to a global solution in general, which is in contrast to the case of the K-dV equation. Nevertheless, the dispersion mechanism of our equations is able to sustain nonconstant travelling waves, which is impossible in the case of Burgers' equation (without dissipation). We discuss the equations in more general forms and derive the properties of our model equations from the general results.



The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

ON THE MODEL EQUATIONS WHICH DESCRIBE NONLINEAR
WAVE MOTIONS IN A ROTATING FLUID

Jong Uhn Kim

0. Introduction

The purpose of this paper is to report some results on qualitative properties of the two model equations which arise in the theory of long waves in a rotating fluid:

$$(0-1) \quad u_t + au_x + uu_x + \lambda \frac{\partial^3}{\partial x^3} \int_{-\infty}^{\infty} \frac{u(\eta, t)}{\sqrt{(x-\eta)^2 + B^2}} d\eta = 0,$$

$$(0-2) \quad u_t + au_x + uu_x - \lambda \frac{\partial}{\partial t} \frac{\partial^2}{\partial x^2} \int_{-\infty}^{\infty} \frac{u(\eta, t)}{\sqrt{(x-\eta)^2 + B^2}} d\eta = 0.$$

Equation (0-1) was derived by Leibovich [9]. This equation describes axially symmetric long wave motions of small amplitude in inviscid, incompressible, rotating fluids which are radially infinite. Here $\lambda \neq 0$, $B > 0$ and a are real constants, t is the time variable and x is the axial coordinate.

When r denotes the radial coordinate, the wave disturbance stream function $\psi(r, x, t)$ is assumed to be of the form $\psi(r, x, t) = \epsilon \phi(r) u(x, t)$ and $u(x, t)$ satisfies (0-1); for details, see [9]. Equation (0-2) is obtained from (0-1) by exploiting the zero-order equivalence of $\frac{\partial}{\partial x}$ and $-\frac{\partial}{\partial t}$ (see BBM [2]).

It is interesting to compare Equation (0-1) with Burgers' equation (without dissipation) and with the K-dV equation:

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$$(0-3) \quad u_t + \alpha u_x + uu_x = 0$$

$$(0-4) \quad u_t + \alpha u_x + uu_x + u_{xxx} = 0.$$

Equation (0-3) is obtained from (0-1) by deleting the dispersion term and Equation (0-4) has a dispersion term which is stronger than that of (0-1); see BBM [2]. Now we consider some basic mathematical questions associated with an evolution equation:

(Q1) Does there exist a unique local solution of the initial value problem in a reasonably smooth function space?

(Q2) Can this local solution be defined globally in time in the same function space?

(Q3) Does the equation admit a nonconstant continuous travelling wave solution?

The main results of the present paper are the answers to these questions; see Theorem 1.1, Theorem 2.1, Theorem 3.4 and Theorem 3.6. Actually we discuss more general equations which include (0-1) and (0-2). The qualitative theory for Equation (0-4) has been developed by many authors and the results are well-known. With regard to the above questions, we compare the equations as follows:

	(0-1), (0-2)	(0-3)	(0-4)
(Q1)	Yes	Yes	Yes
(Q2)	No	No	Yes
(Q3)	Yes	No	Yes

This summary will be explained in details in the subsequent sections.

Notations. The standard notations $\partial_t, \partial_x, f_t, f_x$ are used for

$\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial f}{\partial t}, \frac{\partial f}{\partial x}$, respectively. For a given function g , its Fourier

transform is denoted by \hat{g} and we employ the standard definition:

$$\hat{g}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-ix\xi} dx.$$

$H^s, s > 0$, stands for the set of all real-valued functions g in $L^2(\mathbb{R})$

such that $\int_{-\infty}^{\infty} (1+|\xi|^2)^s |\hat{g}(\xi)|^2 d\xi < \infty$; and the norm $\|\cdot\|_{H^s}$ is taken to be

$\|g\|_{H^s}^2 = \int_{-\infty}^{\infty} (1+|\xi|^2)^s |\hat{g}(\xi)|^2 d\xi$. $H^1(0,p)$ denotes the set of all real-valued functions in $L^2(0,p)$ whose first order derivative is also in $L^2(0,p)$. The

elements of $C_0(\mathbb{R})$ are continuous functions in \mathbb{R} which vanish at infinity.

1. Local Existence of Solutions

Under some physical hypotheses, the general model equations for long waves are given by either

$$(1-1) \quad u_t + \alpha u_x + uu_x + \partial_x P_1 u = 0$$

or

$$(1-2) \quad u_t + \alpha u_x + uu_x + \partial_t P_2 u = 0,$$

where P_1 and P_2 are pseudo-differential operators. In the derivation of such equations, the operators P_i , $i=1,2$, are usually determined from the dispersion relation in terms of their symbols $\hat{P}_i(\xi)$, $i=1,2$ (see [1], [2], [9]). Hence, we assume that the symbols $\hat{P}_i(\xi)$, $i=1,2$, are given in advance to define the operators P_i , $i=1,2$. For (0-1), (0-2), the corresponding operators are given by

$$(1-3) \quad -\hat{P}_1(\xi) = \hat{P}_2(\xi) = \Lambda \sqrt{\frac{2}{\pi}} \xi^2 K_0(B|\xi|),$$

where $K_0(\cdot)$ is the modified Bessel function of the second kind of order zero. In this section, we establish the local existence of solutions to (1-1), (1-2) in the case where the symbols $\hat{P}_i(\xi)$, $i=1,2$, satisfy the following conditions:

$$(1-4) \quad \hat{P}_i(\xi), \quad i=1,2, \quad \text{are even, real-valued functions,}$$

$$(1-5) \quad \xi \hat{P}_i(\xi) \in L^\infty, \quad i=1,2;$$

$$(1-6) \quad \text{for some } -1 < \delta < M < \infty, \quad \delta < \hat{P}_2(\xi) < M \text{ holds for almost all } \xi \in \mathbb{R}.$$

Since $K_0(|\xi|)$ behaves like $-\log|\xi|$ for small ξ and like $\sqrt{\frac{\pi}{2|\xi|}} e^{-|\xi|}$ for large ξ , it is easy to see that (1-4), (1-5) are satisfied by (1-3), and that (1-6) is satisfied when $\lambda > \sqrt{\frac{\pi}{2}} \frac{-1}{\max_{\xi \in R} \xi^2 K_0(B|\xi|)}$. Now we state the

local existence theorem:

Theorem 1.1. Suppose the conditions (1-4) to (1-6) hold. Let

$s > 3/2$ and $u_0(x) \in H^s$. Then, there is a positive number T depending on $\|u_0\|_{H^s}$ such that (1-1), (1-2) have unique solutions in $C([0, T]; H^s) \cap C^1([0, T]; H^{s-1})$ satisfying $u(x, 0) = u_0(x)$.

(proof) Let $s > \frac{3}{2}$ be given. By virtue of (1-4) and (1-5), it is obvious that $f \mapsto \partial_x P_1 f$ is a continuous mapping from H^s into itself satisfying:

$$(1-7) \quad \|\partial_x P_1 f\|_{H^s} \leq \|\hat{\xi} P_1(\xi)\|_{L^\infty} \|f\|_{H^s}, \quad \text{for all } f \in H^s,$$

$$(1-8) \quad \|\partial_x P_1 f - \partial_x P_1 g\|_{L^2} \leq \|\hat{\xi} P_1(\xi)\|_{L^\infty} \|f - g\|_{L^2}, \quad \text{for all } f, g \in H^s.$$

Following the notations in Kato [6], we take $X = L^2$, $Y = H^s$. Then, by means of (1-7), (1-8), we find that Equation (1-1) is a special version of Example 8.1 in [6]; and so, we can employ Kato's result directly to obtain a unique solution in $C([0, T]; H^s) \cap C^1([0, T]; H^{s-1})$ for some $T > 0$. Next we rewrite Equation (1-2) in the form:

$$(1-9) \quad u_t + au_x + uu_x + P_3(au_x + uu_x) = 0,$$

where $\hat{P}_3(\xi) = -\hat{P}_2(\xi)/(1+\hat{P}_2(\xi))$. By taking account of the hypotheses on $\hat{P}_2(\xi)$, it can be easily shown that if $u \in C([0, T]; H^s) \cap C^1([0, T]; H^{s-1})$, $s > \frac{3}{2}$, then (1-2) and (1-9) are equivalent, that is, a solution of (1-9) in the above function space is also a solution of (1-2) and vice versa. The

conditions (1-4) to (1-6) imply that $\hat{\xi}P_3(\xi) \in L^\infty$ and $f \mapsto P_3 f_x$ is a continuous mapping from H^s into itself. The mapping $(f, g) \mapsto fg$ is a continuous bilinear mapping from $H^\sigma \times H^\sigma$ into H^σ , provided $\sigma > \frac{1}{2}$, and hence, it follows that for all $f \in H^s$,

$$(1-10) \quad \|P_3(\alpha f_x + f f_x)\|_{H^s} < M \|\hat{\xi}P_3(\xi)\|_{L^\infty} (\|f\|_{H^s} + \|f\|_{H^s}^2)$$

and for all $f, g \in H^s$,

$$(1-11) \quad \|P_3(\alpha f_x + f f_x) - P_3(\alpha g_x + g g_x)\|_{L^2} < M \|\hat{\xi}P_3(\xi)\|_{L^\infty} \{ \|f+g\|_{H^s} \|f-g\|_{L^2} + \|f-g\|_{L^2}^2 \},$$

where $M > 0$ depends only on α . By taking $X = L^2$, $Y = H^s$, and using (1-10), (1-11), it is easily seen that (1-9) is also a special version of Example 8.1 in [6]; and the local existence of solutions to (1-9) follows immediately.

2. Formation of Singularities

Since local solutions have been obtained in $C([0, T]; H^s)$, $s > \frac{3}{2}$, we shall examine the possibility of extending a local solution to a global solution. It turns out that singularities develop in local solutions under the additional assumptions and thus, a local solution cannot be extended globally in time in general. In the present section, we assume not only (1-4) to (1-6), but also the following conditions:

$$(2-1) \quad \xi^2 \hat{P}_1(\xi) \in L^2,$$

$$(2-2) \quad \xi^2 \hat{P}_2(\xi) \in L^1 \cap L^2.$$

Obviously, (2-1), (2-2) are satisfied by (1-3). The main result of this section is:

Theorem 2.1 Under the above assumptions, the local solutions of (1-1), (1-2) in $C([0, T]; H^s)$, $s > \frac{3}{2}$, cannot be extended globally in time in the same function space if the initial function $u_0(x) \in H^s$ satisfies:

$$(2-3) \quad \frac{\lambda}{\mu} < \kappa_1, \text{ for Equation (1-1);}$$

$$(2-4) \quad \frac{\lambda + \lambda^2}{\mu^2} < \kappa_2, \text{ for Equation (1-2),}$$

where $\lambda = \|u_0\|_{L^2}^2 > 0$, $\mu = -\min_{x \in \mathbb{R}} u_0(x)$ and κ_1 (resp. κ_2) is a positive constant depending only on P_1 (resp. P_2). As a consequence, we have the following

Theorem 2.2 Let $s_1 < 3$ be given. Then, for any $\varepsilon > 0$, there are initial data $u_0(x)$ in H^{s_1} such that $\|u_0\|_{H^{s_1}} < \varepsilon$ and no global solution

satisfying the initial condition $u(x,0) = u_0(x)$ exists in $C([0,\infty); H^{s_2})$, for any $s_2 > \frac{3}{2}$.

(proof) We may take $\frac{3}{2} < s_1 < 3-\delta$ for some $\delta > 0$. Let f be any

(nontrivial) function in H^{s_1} and define $f_\epsilon(x) = \epsilon^{\frac{5}{2}-\delta} f(\frac{x}{\epsilon})$. Let

$\beta = \|f(\cdot)\|_{L^2}^2$, $r = -\min_{x \in R} f_x(x)$ and $\rho = \|f(\cdot)\|_{H^{s_1}}^2$. Then $\|f_\epsilon(\cdot)\|_{L^2}^2 = \epsilon^{3-\delta} \beta$,

$\min_{x \in R} f_{\epsilon x}(x) = -\epsilon^{\frac{3}{2}-\delta} r$ and $\|f_\epsilon(\cdot)\|_{H^{s_1}}^2 < 2^{\frac{s_1}{2}-3-\delta} \epsilon^{\frac{s_1}{2}-3-\delta} \beta + 2^{\frac{s_1}{2}-3-s_1-\delta} \epsilon^{\frac{s_1}{2}-3-s_1-\delta} \rho$. By taking ϵ sufficiently small, $f_\epsilon(\cdot)$ is a desired initial function satisfying (2-3) and (2-4).

To prove Theorem 2.1, we need the following lemma:

Lemma 2.3 Suppose (1-1), (1-2) have solutions $u_1(x,t)$, $u_2(x,t)$, respectively, in $C([0,T); H^s)$, $s > 3/2$, satisfying $u_i(x,0) = u_0(x) \in H^s$, $i=1,2$. Then, it holds that

$$(2-5) \quad \|u_1(x,t)\|_{L^2}^2 = \|u_0(x)\|_{L^2}^2, \text{ for all } t \in [0,T),$$

$$(2-6) \quad \|u_2(x,t)\|_{L^2}^2 \leq M \|u_0(x)\|_{L^2}^2, \text{ for all } t \in [0,T),$$

where M is a positive constant depending only on P_2 .

(proof) Combined with (1-1), (1-2), $u(x,t) \in C([0,T); H^s)$ implies

$u(x,t) \in C^1([0,T); H^{s-1})$. Multiplying both sides of Equations (1-1), (1-2) by $u_1(x,t)$, $u_2(x,t)$, respectively, and integrating over R , we obtain

$$(2-7) \quad \frac{d}{dt} \int_{-\infty}^{\infty} u_1^2(x,t) dx = 0,$$

$$(2-8) \quad \frac{d}{dt} \int_{-\infty}^{\infty} (1+P_2(\xi)) |\hat{u}_2(\xi,t)|^2 d\xi = 0,$$

since

$$(2-9) \quad \int_{-\infty}^{\infty} u_{1x} u_1 dx = 0, \quad \int_{-\infty}^{\infty} (u_1 u_{1x}) u_1 dx = 0, \quad i=1,2;$$

$$(2-10) \quad \int_{-\infty}^{\infty} (P_1 u_{1x}) u_1 dx = \int_{-\infty}^{\infty} i \xi \hat{P}_1(\xi) |\hat{u}_1(\xi, t)|^2 d\xi = 0.$$

(Notice that u_1 is a real-valued function and hence, $|\hat{u}_1(\xi, t)|^2$ is an even function of ξ .) (2-5) is implied by (2-7) and, by making use of (2-8) and (1-6), (2-6) is easily deduced

Now we proceed to:

(proof of Theorem 2.1) With the aid of (1-4) to (1-6) and (2-1), (2-2), we can express $P_1 f_x$ and $P_3 f_x$ (defined in the preceding section) for $f \in H^{\sigma}$, $\sigma > 1$, as follows:

$$(2-11) \quad P_i f_x = \int_{-\infty}^{\infty} G_i(x-y) f(y) dy, \quad i=1,3,$$

where G_i , $i=1,3$, satisfy

$$(2-12) \quad G_1 \in L^2, \quad G_{1x} \in L^2,$$

$$(2-13) \quad G_3 \in L^2 \cap C_0, \quad G_{3x} \in L^2 \cap C_0.$$

We assume that for given nontrivial $u_0(x)$ in H^s , $s > \frac{3}{2}$, there is a global solution $u(x, t)$ of (1-1) in $C([0, \infty); H^s)$ which, combined with (1-1), implies $u \in C^1(\mathbb{R} \times [0, \infty))$. Following the classical theory of the characteristics, we solve the initial value problem for each fixed $\eta \in \mathbb{R}$:

$$(2-14) \quad \frac{dx(t, \eta)}{dt} = \alpha + u(x(t, \eta), t),$$

$$(2-15) \quad x(0, \eta) = \eta.$$

Observing that $u(x, t)$ is uniformly bounded on $\mathbb{R} \times [0, T]$, for each $T > 0$, we can conclude that $x(t, \eta) \in C^1([0, \infty) \times \mathbb{R})$ and that $x(t, \eta_2) > x(t, \eta_1)$ for all $t > 0$, provided $\eta_2 > \eta_1$ (see Hale [5]). With this $x(t, \eta)$, $u(x(t, \eta), t)$ is regarded as a function of t for each η and satisfies, for

each fixed η ,

$$(2-16) \quad \frac{d}{dt} u(x(t, \eta), t) = - \int_{-\infty}^{\infty} G_1(x(t, \eta) - y) u(y, t) dy, \quad \text{for all } t > 0,$$

and

$$(2-17) \quad u(x(0, \eta), 0) = u_0(\eta).$$

Since $u_0(\eta) \not\equiv 0$, there are $\mu > 0$ and $\eta_1 < \eta_2$ such that

$$(2-18) \quad \frac{u_0(\eta_2) - u_0(\eta_1)}{\eta_2 - \eta_1} = -\mu.$$

Let us set $\omega = u_0(\eta_1) - u_0(\eta_2) > 0$, $\eta_2 - \eta_1 = l > 0$. From (2-16), (2-17), it follows that for all $t > 0$,

$$(2-19) \quad u(x(t, \eta_1), t) - u(x(t, \eta_2), t) = u_0(\eta_1) - u_0(\eta_2) - \int_0^t d\tau \int_{-\infty}^{\infty} \{G_1(x(\tau, \eta_1) - y) - G_1(x(\tau, \eta_2) - y)\} u(y, \tau) dy,$$

which, together with Lemma 2.3 and (2-12), implies

$$(2-20) \quad u(x(t, \eta_1), t) - u(x(t, \eta_2), t) > \omega - \int_0^t d\tau M_1 \lambda |x(\tau, \eta_1) - x(\tau, \eta_2)|,$$

for all $t > 0$, and

$$(2-21) \quad |u(x(t, \eta_1), t) - u(x(t, \eta_2), t)| \leq \omega + \int_0^t d\tau M_1 \lambda |x(\tau, \eta_1) - x(\tau, \eta_2)|,$$

for all $t > 0$, where $M_1 = \|G_{1x}\|_{L^2}$ and $\lambda = \|u_0(\eta)\|_{L^2}$. But, (2-14) and

(2-15) yield

$$(2-22) \quad |x(\tau, \eta_1) - x(\tau, \eta_2)| \leq l + \int_0^{\tau} d\zeta |u(x(\zeta, \eta_1), \zeta) - u(x(\zeta, \eta_2), \zeta)|.$$

Consequently, for all $0 < t < \frac{1}{\mu}$, we have

$$(2-23) \quad \begin{aligned} & |u(x(t, \eta_1), t) - u(x(t, \eta_2), t)| \\ & \leq \omega + \int_0^t d\tau M_1 \lambda \left\{ \ell + \int_0^\tau d\zeta |u(x(\zeta, \eta_1), \zeta) - u(x(\zeta, \eta_2), \zeta)| \right\}, \end{aligned}$$

by using $\omega = \ell\mu$,

$$\begin{aligned} & \leq \omega \left(1 + \frac{M_1 \lambda}{\mu^2} \right) + M_1 \lambda \int_0^t d\zeta (t - \zeta) |u(x(\zeta, \eta_1), \zeta) - u(x(\zeta, \eta_2), \zeta)| \\ & \leq \omega \left(1 + \frac{M_1 \lambda}{\mu^2} \right) + M_1 \lambda \int_0^t d\zeta \left(\frac{1}{\mu} - \zeta \right) |u(x(\zeta, \eta_1), \zeta) - u(x(\zeta, \eta_2), \zeta)|. \end{aligned}$$

By Gronwall's inequality, we arrive at

$$(2-24) \quad |u(x(t, \eta_1), t) - u(x(t, \eta_2), t)| \leq \omega \left(1 + \frac{M_1 \lambda}{\mu^2} \right) e^{M_1 \lambda / 2\mu^2},$$

for all $0 < t < \frac{1}{\mu}$, and hence,

$$(2-25) \quad M_1 \lambda \int_0^{1/\mu} \left(\frac{1}{\mu} - \zeta \right) |u(x(\zeta, \eta_1), \zeta) - u(x(\zeta, \eta_2), \zeta)| d\zeta \leq M_1 \lambda \omega \left(1 + \frac{M_1 \lambda}{\mu^2} \right) \frac{1}{2\mu^2} e^{M_1 \lambda / 2\mu^2}.$$

In the mean time, substituting (2-22) into (2-20) we have, for all

$$0 < t < \frac{1}{\mu},$$

$$\begin{aligned} & u(x(t, \eta_1), t) - u(x(t, \eta_2), t) > \omega \left(1 - \frac{M_1 \lambda}{\mu^2} \right) \\ & - M_1 \lambda \int_0^t d\tau \int_0^\tau d\zeta |u(x(\zeta, \eta_1), \zeta) - u(x(\zeta, \eta_2), \zeta)| \\ (2-26) \quad & = \omega \left(1 - \frac{M_1 \lambda}{\mu^2} \right) - M_1 \lambda \int_0^t d\zeta (t - \zeta) |u(x(\zeta, \eta_1), \zeta) - u(x(\zeta, \eta_2), \zeta)| \\ & > \omega \left(1 - \frac{M_1 \lambda}{\mu^2} \right) - M_1 \lambda \int_0^{1/\mu} d\zeta \left(\frac{1}{\mu} - \zeta \right) |u(x(\zeta, \eta_1), \zeta) - u(x(\zeta, \eta_2), \zeta)|, \end{aligned}$$

by using (2-25),

$$> \omega \left\{ 1 - \frac{M_1 \lambda}{\mu^2} - \frac{M_1 \lambda}{2\mu^2} \left(1 + \frac{M_1 \lambda}{\mu^2} \right) e^{M_1 \lambda / 2\mu^2} \right\}.$$

Now we choose $\kappa > 0$ such that

$$(2-27) \quad M_1 \left\{ \kappa + \frac{1}{2} \kappa (1 + M_1 \kappa) e^{\frac{1}{2} M_1 \kappa} \right\} < \frac{1}{2},$$

$$(2-28) \quad M_1 \kappa e^{\frac{1}{2} M_1 \kappa} < \frac{1}{2}.$$

Then,

$$(2-29) \quad \frac{\lambda}{\mu^2} < \kappa$$

implies

$$(2-30) \quad u(x(t, \eta_1), t) - u(x(t, \eta_2), t) > \frac{1}{2} \omega, \quad \text{for all } 0 < t < \frac{1}{\mu}.$$

For the remainder of proof, we assume (2-29) holds. We proceed to estimate

$x(t, \eta_2) - x(t, \eta_1)$. Combining (2-14) to (2-17), it is apparent that

$$(2-31) \quad x(t, \eta) = \eta + \int_0^t d\tau \{ \alpha + u_0(\eta) - \int_0^\tau d\zeta \int_{-\infty}^\infty G_1(x(\zeta, \eta) - y) u(y, \zeta) dy \},$$

for all $t > 0$, $\eta \in \mathbb{R}$, and thus,

$$(2-32) \quad \begin{aligned} x(t, \eta_2) - x(t, \eta_1) = \\ l - \omega t - \int_0^t d\tau \int_0^\tau d\zeta \left\{ \int_{-\infty}^\infty (G_1(x(\zeta, \eta_2) - y) - G_1(x(\zeta, \eta_1) - y)) u(y, \zeta) dy \right\}. \end{aligned}$$

Set $f(t) = x(t, \eta_2) - x(t, \eta_1)$; then, $f(t) > 0$ for all $t > 0$. From (2-32),

we derive the estimate:

$$(2-33) \quad \begin{aligned} f(t) &\leq l - \omega t + \int_0^t d\tau \left\{ M_1 \lambda \int_0^\tau f(\zeta) d\zeta \right\} \\ &= l - \omega t + M_1 \lambda \int_0^t (t - \zeta) f(\zeta) d\zeta \\ &< l - \omega t + M_1 \lambda \int_0^t \left(\frac{1}{\mu} - \zeta \right) f(\zeta) d\zeta, \quad \text{for all } 0 < t < \frac{1}{\mu}. \end{aligned}$$

By the generalized Gronwall inequality (see Hale [5]),

$$\begin{aligned}
 (2-34) \quad f(t) &< 1 - \omega t + M_1 \lambda \int_0^t \left(\frac{1}{\mu} - \zeta \right) (1 - \omega \zeta) e^{M_1 \lambda / 2 \mu^2 \zeta} d\zeta \\
 &= 1 - \omega t + M_1 \lambda e^{M_1 \lambda / 2 \mu^2} \left(\frac{\omega}{\mu^2} t - \frac{\omega}{\mu} t^2 + \frac{1}{3} \omega t^3 \right),
 \end{aligned}$$

for all $0 < t < \frac{1}{\mu}$.

In particular,

$$(2-35) \quad f\left(\frac{1}{\mu}\right) < \frac{1}{3} M_1 \lambda e^{M_1 \lambda / 2 \mu^2} \frac{\omega}{\mu^3},$$

from which we deduce, using (2-28), (2-29),

$$(2-36) \quad \frac{u\left(x\left(\frac{1}{\mu}, \eta_2\right), \frac{1}{\mu}\right) - u\left(x\left(\frac{1}{\mu}, \eta_1\right), \frac{1}{\mu}\right)}{x\left(\frac{1}{\mu}, \eta_2\right) - x\left(\frac{1}{\mu}, \eta_1\right)} < -3\mu.$$

Hence, for some $y_2 > y_1$,

$$(2-36) \quad \frac{u\left(y_2, \frac{1}{\mu}\right) - u\left(y_1, \frac{1}{\mu}\right)}{y_2 - y_1} = -3\mu$$

Let us summarize what we have obtained so far: If we suppose that

$$\begin{aligned}
 u(x, t) &\in C([0, \infty); H^s), \quad s > \frac{3}{2}, \quad \frac{u(x_2, 0) - u(x_1, 0)}{x_2 - x_1} = -\mu < 0, \quad x_2 > x_1 \quad \text{and} \\
 \frac{\|u(x, 0)\|_{L^2}^2}{\mu^2} &< \kappa \quad (\text{defined by (2-27), (2-28)}), \quad \text{then at time } t = \frac{1}{\mu}, \\
 \frac{u(y_2, t) - u(y_1, t)}{y_2 - y_1} &= -3\mu, \quad \text{for some } y_2 > y_1.
 \end{aligned}$$

Recalling that $\|u(x, t)\|_{L^2}^2$ is constant for all $t > 0$, taking $\frac{1}{\mu}$ as our initial time and repeating the above process, we arrive at

$$(2-38) \quad \frac{u\left(z_2, \frac{1}{\mu} + \frac{1}{3\mu}\right) - u\left(z_1, \frac{1}{\mu} + \frac{1}{3\mu}\right)}{z_2 - z_1} = -9\mu,$$

for some $z_2 > z_1$. By indefinite iteration, we conclude that as t approaches $\frac{1}{u} \left(1 + \frac{1}{3} + \frac{1}{3^2} + \dots\right) = \frac{3}{2} \frac{1}{u}$, $\min_{x \in R} u(x, t)$ tends to $+\infty$, which contradicts our assumption that $u(x, t) \in C([0, \infty); H^s)$, $s > \frac{3}{2}$. This completes the proof for Equation (1-1). As mentioned in the preceding section, Equations (1-2) and (1-9) are equivalent provided

$u(x, t) \in C([0, T); H^s)$, $s > 3/2$; hence, we consider (1-9) to prove the assertion for (1-2). By virtue of (2-13), we derive that for all $f \in L^2(R)$,

$$(2-39) \quad \left| \int_{-\infty}^{\infty} \{G_3(x_1 - y) - G_3(x_2 - y)\} f(y) dy \right| < |x_1 - x_2| \|G_{3x}\|_{L^2} \|f\|_{L^2}$$

and

$$(2-40) \quad \left| \int_{-\infty}^{\infty} \{G_3(x_1 - y) - G_3(x_2 - y)\} |f(y)|^2 dy \right| < |x_1 - x_2| \|G_{3x}\|_{L^\infty} \|f\|_{L^2}^2.$$

Using these inequalities and Lemma 2.3, and going through the same procedure as above, we can arrive at a similar conclusion for Equation (1-9); we omit the details.

Seliger [10] also considered some model equations of nonlinear wave motions. However, the method in [10] is not applicable to our problem. The main reason is that $u_{xx}(x, t)$ may not be defined as an ordinary function under our assumption. Even if all the assumptions in [10] were met, a result analogous to Theorem 2.2 cannot be inferred from the result of [10]. Finally, we recall some known results on the solutions of (0-3) and (0-4). For a given initial function in H^s , $s > \frac{3}{2}$, (0-3) and (0-4) have unique local solutions in $C([0, T); H^s)$, $s > \frac{3}{2}$; see Kato [6], [7]. It is known that a local solution of (0-4) in $C([0, T); H^s)$ can be extended globally in time provided $s > 2$, while (0-3) does not admit any global solution in $C([0, \infty); H^s)$, $s > \frac{3}{2}$, unless the initial function is identically zero.

3. Existence of travelling wave solutions

We shall seek solutions of (1-1), (1-2) in the form $\phi(x-ct)$, $\phi \neq \text{constant}$. For this purpose we consider the following equations with $c \neq a$:

$$(3-1) \quad \phi - \frac{1}{c-a} P_1 \phi = \frac{1}{2} \frac{1}{c-a} \phi^2,$$

$$(3-2) \quad \phi - \frac{c}{a-c} P_2 \phi = \frac{1}{2} \frac{1}{c-a} \phi^2.$$

Let us suppose that P_1, P_2 are convolution operators with a kernel in $L^1(\mathbb{R})$ and that $\phi(\cdot)$ is a solution of (3-1) (resp. (3-2)) in $L^\infty(\mathbb{R})$. Then, $\phi(x-ct)$ is a solution (in the distribution sense) of (1-1) (resp. (1-2)) provided uu_x is interpreted as $\left(\frac{1}{2}u^2\right)_x$. Therefore, we try to find solutions of (3-1), (3-2) under the following assumptions on P_1 and P_2 :

(3-3) $\hat{P}_i(\xi)$, $i=1,2$, are even, real-valued functions;

(3-4) $\hat{P}_i(\xi) \in L^1 \cup L^2$ and $\frac{d}{d\xi} \hat{P}_i(\xi) \in L^2$, $i = 1,2$;

(3-5) there exist $p > 0$ and $s \neq 0$, $\hat{P}_1(0)$, such that the set $\{\xi: \hat{P}_1(\xi) = s\} \cap \{2\pi k/p: k=0,1,2,\dots\}$ is not empty and consists of odd number of points;

(3-6) there exist $p > 0$ and $s \neq \frac{-1}{\sqrt{2\pi}}$, 0 , $\hat{P}_2(0)$, such that set $\{\xi: \hat{P}_2(\xi) = s\} \cap \{\frac{2\pi k}{p}: k=0,1,2,\dots\}$ is not empty and consists of odd number of points.

Theorem 3.1 Under the assumptions (3-3) to (3-5), there are nonconstant, periodic solutions to (3-1) for suitable $c \neq a$.

Theorem 3.2 Suppose that $a \neq 0$ and that (3-3), (3-4) and (3-6) hold. Then, (3-2) admits nonconstant periodic solutions for suitable $c \neq a$.

(proof of Theorem 3.1) Let us write (3-1) as

$$(3-7) \quad \phi - \lambda P_1 \phi = \frac{1}{2} \lambda \phi^2,$$

where λ is a parameter. By virtue of (3-4), we can define a function $F(\cdot)$ such that $\hat{F}(\xi) = \hat{P}_1(\xi)$; then, F is an even function in $L^1(\mathbb{R})$ (see [8]). Consequently, P_1 is defined by

$$(3-8) \quad (P_1 \phi)(x) = \int_{-\infty}^{\infty} F(x-y) \phi(y) dy, \text{ for all } \phi \in L^\infty(\mathbb{R}).$$

Next we define

$$(3-9) \quad F_p(x) = \sum_{k=-\infty}^{\infty} F(x+kp),$$

with p which appeared in (3-5). Then the series converges absolutely at almost all x and $F_p(x)$ is a p -periodic, even function. It is easy to see that $F_p(x)$ is integrable over $[0, p]$ and can be expanded in trigonometric functions:

$$(3-10) \quad \begin{aligned} F_p(x) &\sim \sum_{k=-\infty}^{\infty} \frac{\sqrt{2\pi}}{p} \hat{P}_1\left(\frac{2\pi k}{p}\right) e^{i \frac{2\pi k}{p} x} \\ &\sim \frac{\sqrt{2\pi}}{p} \hat{P}_1(0) + \sum_{k=1}^{\infty} \frac{2\sqrt{2\pi}}{p} \hat{P}_1\left(\frac{2\pi k}{p}\right) \cos\left(\frac{2\pi k}{p} x\right). \end{aligned}$$

Next we define the function space

$$(3-11) \quad S_p = \left\{ \sum_{k=0}^{\infty} a_k \cos\left(\frac{2\pi k}{p} x\right) : a_k \in \mathbb{R}, \sum_{k=0}^{\infty} |a_k|^2 \left\{1 + \left(\frac{2\pi k}{p}\right)^2\right\} < \infty \right\}$$

equipped with the norm $\left(\sum_{k=0}^{\infty} |a_k|^2 \left\{1 + \left(\frac{2\pi k}{p}\right)^2\right\} \right)^{\frac{1}{2}}$ and the corresponding inner product; then, S_p is a real Hilbert space. We also define the operator T_p by

$$(3-12) \quad (T_p \phi)(x) = \int_0^p P_p(x-y) \phi(y) dy, \text{ for each } \phi \in S_p.$$

Then we have

Lemma 3.3 T_p is a compact, self-adjoint operator in S_p and has an eigenvalue $\sqrt{2\pi} s$ of odd multiplicity (recall that s and p appeared in (3-5)).

(proof) Let $\phi = \sum_{k=0}^{\infty} a_k \cos(\frac{2\pi k}{p}x) \in S_p$. Then, from (3-10) and (3-12), it follows that

$$(3-13) \quad T_p \phi \sim \sum_{k=0}^{\infty} \sqrt{2\pi} \hat{P}_1(\frac{2\pi k}{p}) a_k \cos(\frac{2\pi k}{p}x).$$

By the Riemann-Lebesgue Lemma, $\hat{P}_1(\frac{2\pi k}{p})$ converges to 0 as k tends to infinity and we obtain the estimate

$$(3-14) \quad \sum_{k=0}^{\infty} |\hat{P}_1(\frac{2\pi k}{p})|^2 |a_k|^2 \{1 + (\frac{2\pi k}{p})^2\} < \max_k |\hat{P}_1(\frac{2\pi k}{p})|^2 \sum_{k=0}^{\infty} |a_k|^2 \{1 + (\frac{2\pi k}{p})^2\},$$

which implies that the infinite series of (3-13) converges absolutely for all x and is equal to $(T_p \phi)(x)$ for all x . Now it is obvious that T_p is a continuous mapping from S_p into itself and that T_p is self-adjoint in S_p . For each $m > 1$, the operator $T_{p,m}$ defined by

$$(3-15) \quad T_{p,m} \phi = \sum_{k=0}^m \sqrt{2\pi} \hat{P}_1(\frac{2\pi k}{p}) a_k \cos(\frac{2\pi k}{p}x),$$

for $\phi = \sum_{k=0}^{\infty} a_k \cos(\frac{2\pi k}{p}x) \in S_p$, is a compact operator from S_p into itself. Since the operator norm $\|T_p - T_{p,m}\|$ is bounded by $\sqrt{2\pi} \max_{k > m+1} |\hat{P}_1(\frac{2\pi k}{p})|$, we conclude that $\|T_p - T_{p,m}\|$ converges to 0 as m tends to infinity. Thus, T_p is also compact. Finally, combining (3-5) and (3-13), we deduce that $\sqrt{2\pi} s$ is an eigenvalue of odd multiplicity in S_p .

Lemma 3.4 The mapping of $\phi \rightarrow \phi^2$ is a C^∞ -mapping from S_p into itself.

(proof) First we observe that the S_p norm is equivalent to the $H^1(0,p)$ norm and that S_p is a closed subspace of $H^1(0,p)$. Let $\phi = \sum_{k=0}^{\infty} a_k \cos(\frac{2\pi k}{p}x) \in S_p$ and $\phi_m = \sum_{k=0}^m a_k \cos(\frac{2\pi k}{p}x)$. From the identity $\cos\beta \cos\gamma = \frac{1}{2}[\cos(\beta+\gamma) + \cos(\beta-\gamma)]$, we infer that $\phi_m^2 \in S_p$ for each m . In the meantime, it is known that if $f, g \in H^1(0,p)$, then $fg \in H^1(0,p)$ and $\|fg\|_{H^1(0,p)} < M\|f\|_{H^1(0,p)}\|g\|_{H^1(0,p)}$

holds with a constant $M > 0$ independent of f, g . Therefore ϕ_m^2 converges to ϕ^2 in $H^1(0,p)$ as m tends to infinity and we conclude that $\phi^2 \in S_p$. Now it is easy to see that the mapping $\phi \rightarrow \phi^2$ is C^∞ from S_p into itself.

With the aid of Lemmas 3.3, 3.4, we can easily show that all the hypotheses in Theorem A of Westreich [11] are satisfied. Hence, it follows that $(\frac{1}{\sqrt{2\pi s}}, 0) \in R \times S_p$ is a bifurcation point for the equation

$$(3-16) \quad \phi - \lambda T_p \phi = \frac{1}{2} \lambda \phi^2.$$

In other words, for any given $\varepsilon > 0$, there is a nonzero element ϕ_ε in S_p and $\lambda_\varepsilon \in R$ such that $|\lambda_\varepsilon - \frac{1}{\sqrt{2\pi s}}| < \varepsilon$, $\|\phi_\varepsilon\|_{S_p} < \varepsilon$ and $(\lambda_\varepsilon, \phi_\varepsilon)$ satisfies (3-16). It remains to show that for sufficiently small $\varepsilon > 0$, ϕ_ε cannot be a constant function. In fact, when $\phi_\varepsilon \equiv \rho$, for some constant ρ , $0 < |\rho| < \varepsilon$, (3-16) implies

$$(3-17) \quad \rho - \lambda_\varepsilon \sqrt{2\pi} \hat{P}_1(0)\rho = \frac{1}{2} \lambda_\varepsilon \rho^2.$$

If ε is sufficiently small, $1 - \lambda_\varepsilon \sqrt{2\pi} \hat{P}_1(0)$ is bounded away from zero by the condition (3-5) and consequently, (3-17) cannot hold for small $\rho \neq 0$. If $\phi \in S_p$, then ϕ is a p -periodic, continuous function in $L^\infty(R)$ and it holds that

$$(3-18) \quad \int_0^P F_p(x-y)\phi(y)dy = \int_{-\infty}^{\infty} F(x-y)\phi(y)dy.$$

Hence, a solution of (3-16) in S_p is also a solution of (3-1) by choosing suitable $c \neq \alpha$. This completes the proof of Theorem 3.1.

(proof of Theorem 3.2) Suppose that $\alpha \neq 0$ and write (3-2) as

$$(3-19) \quad \phi - \lambda P_2 \phi = \frac{-(\lambda+1)}{2\alpha} \phi^2,$$

where λ is a parameter. The above proof of Theorem 3.1 can be repeated to establish the existence of nonconstant periodic solutions to (3-2). The additional conditions that $s \neq \frac{1}{\sqrt{2\pi}}$ and that $\alpha \neq 0$ are necessary to choose a constant c corresponding to suitable λ , that is, $c \in \mathbb{R}$ can be chosen so that $\frac{c}{\alpha-c} = \lambda$ when $\lambda \neq -1$, $\alpha \neq 0$.

As an application of the above results, we specialize on (0-1) and (0-2), which we rewrite as

$$(3-20) \quad u_t + \alpha u_x + uu_x + \frac{\partial}{\partial x} \int_{-\infty}^{\infty} \Gamma(x-y)u(y,t)dy = 0,$$

$$(3-21) \quad u_t + \alpha u_x + uu_x - \frac{\partial}{\partial t} \int_{-\infty}^{\infty} \Gamma(x-y)u(y,t)dy = 0,$$

where $\Gamma(x) = \lambda \frac{\partial^2}{\partial x^2} \frac{1}{\sqrt{x^2+B^2}}$. This interpretation of (0-1) and (0-2) is admissible since the original model equation was derived in terms of the symbol of the operator (see [9]). Moreover, this form of the equations has an advantage in respect that the integral is well-defined when $u(\cdot, t)$ is merely in L^∞ . Restricting our attention to the case $c > \alpha$, our claim for Equation (3-20) is:

Theorem 3.4 Suppose that $A \neq 0$ and $B > 0$. Then, (3-20) has nonconstant, periodic travelling wave solutions of the form $\phi(x-ct)$, $c > \alpha$. If $A < 0$,

the amplitude of ϕ can be arbitrarily small (with $c - \alpha$ bounded away from zero). If $A > 0$ and $c - \alpha > \varepsilon$ for some $\varepsilon > 0$, then there is no nonconstant, periodic solution of the form $\phi(x-ct)$ with $\|\phi\|_{\infty} < \frac{\varepsilon}{2}$ (proof) We consider Equation (3-1) with $\hat{p}_1(\xi) = -A\sqrt{\frac{2}{\pi}} \xi^2 K_0(B|\xi|)$, where $K_0(\cdot)$ is the modified Bessel function of the second kind of order zero. It is known that $\xi^2 K_0(|\xi|)$ is nonnegative for all ξ and that it behaves like $-\xi^2 \log|\xi|$ for small ξ , and like $\sqrt{\frac{\pi}{2}} |\xi|^{\frac{3}{2}} e^{-|\xi|}$ for large ξ . Moreover, $\max_{\xi > 0} \xi^2 K_0(B|\xi|)$ occurs at a single point, which we denote by ξ_0 ; $1.55/B < \xi_0 < 1.56/B$. Accordingly, the conditions (3-3), (3-4) are satisfied and we choose

$$(3-22) \quad s_0 = -A\sqrt{\frac{2}{\pi}} \xi_0^2 K_0(B\xi_0)$$

and

$$(3-23) \quad p_0 = \frac{2\pi k_0}{\xi_0}, \text{ for any fixed positive integer } k_0,$$

so that the condition (3-5) is satisfied. In fact, the set

$\{\xi: \hat{p}_1(\xi) = s_0\} \cap \{\frac{2\pi k}{p_0}: k=0,1,2,\dots\}$ is a single point ξ_0 and hence, the eigenvalue $\sqrt{2\pi} s_0$ (or $-2A\xi_0^2 K_0(B\xi_0)$) is simple with an eigenfunction $\cos(\xi_0 x)$. Here we can use a theorem of Crandall-Rabinowitz [4] to conclude that (3-1) has solutions

$$\phi(x) = \varepsilon \cos(\xi_0 x) + \varepsilon g(\varepsilon)$$

with $c - \alpha = \sqrt{2\pi} s_0 + h(\varepsilon)$ for sufficiently small $\varepsilon > 0$ where $g(\varepsilon)$ is a C^∞ function of ε into the complement of $\text{span}\{\cos(\xi_0 x)\}$ in S_{p_0} and $h(\varepsilon)$ is a C^∞ function of ε into \mathbb{R} such that $g(0) = 0$, $h(0) = 0$. When $A < 0$, s_0 is positive by (3-22) and thus, the first part of the theorem has been proved. Next let us consider the case $A > 0$, which implies $s_0 < 0$. Since we are interested in solutions of the form

$\phi(x-ct)$, $c > a$, we define a new function Ψ by

$$(3-24) \quad \Psi = -2(c-a) + \phi,$$

and write (3-1) in terms of Ψ :

$$(3-25) \quad \Psi - \frac{1}{a-c} P_1 \Psi = \frac{1}{2} \frac{1}{a-c} \Psi^2,$$

where we have used the fact that P_1 annihilates constant functions since $\hat{P}_1(\xi) = -A\sqrt{\frac{2}{\pi}} \xi^2 K_0(B|\xi|)$. Now we have $a - c$ in place of $c - a$, which is the only difference between (3-1) and (3-25). So we know that (3-25) has solutions $\epsilon \cos(\xi_0 x) + \epsilon g(\epsilon)$ with $a - c = \sqrt{2\pi} a_0 + h(\epsilon)$ for sufficiently small $\epsilon > 0$, where g, h are the same functions as above. Hence we have obtained solutions $\phi(x)$ of (3-1) in the form

$$\phi(x) = 2(2A\xi_0^2 K_0(B\xi_0) - h(\epsilon)) + \epsilon \cos(\xi_0 x) + \epsilon g(\epsilon),$$

with $c - a = 2A\xi_0^2 K_0(B\xi_0) - h(\epsilon)$, which is positive when $A > 0$ and ϵ is sufficiently small. Finally, we suppose that $A > 0$, $c - a > \epsilon > 0$ and that (3-20) has a nonconstant periodic solution $\phi(x-ct)$ in L^∞ with period p . Then, using (3-18), we find that $\phi(x)$ satisfies

$$(3-26) \quad \phi(x) - \frac{1}{c-a} \int_0^p P_p(x-y)\phi(y)dy = \frac{1}{2(c-a)}\phi^2(x) + M,$$

where M is a constant. Let us define

$$(3-27) \quad \sigma(x) = \phi(x) - \frac{1}{p} \int_0^p \phi(y)dy$$

and write (3-26) in terms of $\sigma(x)$:

$$(3-28) \quad \sigma(x) - \frac{1}{c-a} \int_0^P F_p(x-y) \sigma(y) dy = \frac{1}{2(c-a)} \sigma^2(x) + \frac{1}{p(c-a)} \left(\int_0^P \phi(y) dy \right) \sigma(x) + \tilde{M},$$

where \tilde{M} is a constant and we have used the fact that $\int_0^P F_p(x-y) dy = 0$ (notice that $\hat{P}_1(0) = 0$). Multiplying both sides of (3-28) by $\sigma(x)$ and integrating over $(0, p)$, we obtain

$$(3-29) \quad \int_0^P \sigma^2(x) dx < \frac{1}{c-a} \left\{ \frac{1}{2} \|\sigma\|_{L^\infty}^2 + \frac{1}{p} \left| \int_0^P \phi(y) dy \right| \right\} \int_0^P \sigma^2(x) dx < \frac{1}{c-a} 2 \|\phi\|_{L^\infty} \int_0^P \sigma^2(x) dx,$$

since $\int_0^P \sigma(x) dx = 0$ and $\int_0^P \left(\int_0^P F_p(x-y) \phi(y) dy \right) \phi(x) dx =$

$$\sqrt{2\pi p} \hat{P}_1(0) a_0^2 + \sum_{k=1}^{\infty} \sqrt{2\pi p} \hat{P}_1\left(\frac{2\pi k}{p}\right) (a_k^2 + b_k^2) = - \sum_{k=1}^{\infty} A_p \left(\frac{2\pi k}{p}\right)^2 K_0\left(\frac{2\pi k}{p}\right) (a_k^2 + b_k^2) < 0,$$

where $\phi(x) \sim \sum_{k=0}^{\infty} (a_k \cos \frac{2\pi k}{p} x + b_k \sin \frac{2\pi k}{p} x)$. Since ϕ is nonconstant,

$$\int_0^P \sigma^2(x) dx > 0 \text{ and hence, it follows from (3-29) that } \|\phi\|_{L^\infty} > \frac{c-a}{2} > \frac{\varepsilon}{2}. \text{ This}$$

concludes the proof of Theorem 3.4.

Remark 3.5 We note that there are infinitely many s, p such that

$\{\xi: \hat{P}_1(\xi) = s\} \cap \{\frac{2\pi k}{p}: k=0, 1, 2, \dots\}$ is a single point; we could have chosen

other points rather than s_0, p_0 in the above proof. Now we elaborate on this fact. When $P_1(\xi) = -A\sqrt{\frac{2}{\pi}} \xi^2 K_0(B|\xi|)$, $\hat{P}_1(\xi)$ is monotonically increasing on $[0, \xi_0]$ and monotonically decreasing on (ξ_0, ∞) or vice versa depending on A . Therefore, it is easy to show that the set

$\{s: \exists p > 0, \exists s, p \text{ satisfy (3-5)}\}$ is dense in the interval $(0, s_0)$ (or $(s_0, 0)$). Suppose $A < 0$. Let s_1 be any number in $(0, s_0)$ and $\xi_1 < \xi_2$ be two roots of $\hat{P}_1(\xi) = s_1$. If (3-5) is not satisfied by s_1 , we infer that if $\xi_1 = 2\pi k_1/p$ for some $p > 0$ and integer $k_1 > 0$, then there is a positive integer k_2 so that $\xi_2 = 2\pi k_2/p$. Suppose this is the case. Set $s_1^* = s_1 - \delta > 0$, for small $\delta > 0$. Then, $\xi_1 = 2\pi k_1/p, \xi_2 = 2\pi k_2/p$ for some $p > 0$ and integers $k_1, k_2 > 0$. Let $\xi_1^* < \xi_2^*$ be roots of

$\hat{P}_1(\xi) = s_1^*$ and choose p^* so that $\xi_1^* = 2\pi k_1/p^*$. Obviously,
 $\xi_1^* < \xi_1 < \xi_2 < \xi_2^*$. Thus, $p^* > p$ and $2\pi k_2/p^* < 2\pi k_2/p = \xi_2$. If δ is
sufficiently small, then p^*, ξ_2^* are so close to p, ξ_2 , respectively, that
there is no integer $k_2^* > 0$ such that $\xi_2^* = 2\pi k_2^*/p^* > 2\pi k_2/p = \xi_2$. Hence,
 $s_1 - \delta$ may be taken to satisfy (3-5). The proof for the case $A > 0$ is
similar.

Now we turn our attention to Equation (3-21), for which our assertion is:
Theorem 3.6 Suppose $\alpha > 0$, $A \neq 0$ and $B > 0$. Then, (3-21) has nontrivial,
periodic travelling wave solutions of the form $\phi(x-ct)$, $c > \alpha$. If $A < 0$,
the amplitude of ϕ can be arbitrarily small (with $c-\alpha$ bounded away from
zero). If $A > 0$ and $c - \alpha > \varepsilon$, then there is no nonconstant, periodic
solution of the form $\phi(x-ct)$ with $\|\phi\|_{L^\infty} < \frac{\varepsilon}{2}$.
(proof) We consider Equation (3-2) with $\hat{P}_2(\xi) = A\sqrt{\frac{2}{\pi}} \xi^2 K_0(B|\xi|)$, where the
bifurcation parameter λ is related to c by $\frac{c}{\alpha-c} = \lambda$. Hence, $c > \alpha$ holds
if and only if $\lambda < -1$. Suppose $A < 0$. With the aid of Remark 3.5, we can
find $\xi_1 > 0$, $p_1 > 0$ and integer $k_1 > 0$ such that $\xi_1 = 2\pi k_1/p_1$,
 $-1 < 2A\xi_1^2 K_0(B\xi_1) < 0$ and such that $s_1 = A\sqrt{\frac{2}{\pi}} \xi_1^2 K_0(B\xi_1)$, p_1 satisfy the
condition (3-6). By repetition of the previous argument, we derive that (3-2)
has solutions

$$\varepsilon \cos(\xi_1 x) + \varepsilon g_1(\varepsilon)$$

with $\lambda = \frac{1}{\sqrt{2\pi} s_1} + h_1(\varepsilon)$ for sufficiently small $\varepsilon > 0$, where $g_1(\varepsilon)$ is a
 C^∞ mapping of ε into the complement of $\{\cos(\xi_1 x)\}$ in S_{p_1} and $h_1(\varepsilon)$
is a C^∞ -mapping of ε into \mathbb{R} such that $g_1(0) = 0$, $h_1(0) = 0$. Now we can
choose $c > \alpha$ such that $\frac{c}{\alpha-c} = \lambda < -1$. For the case $A > 0$, we again use
Remark 3.5 after introducing $\psi \stackrel{\text{def}}{=} \frac{2\alpha}{\lambda+1} + \phi$ as in the proof of Theorem 3.4.
The last assertion is also proved as before and we shall omit the details.

Remark 3.7 It is well known that the K-dV equation admits both solitary wave solutions and periodic travelling wave solutions. For Equations (3-20), (3-21), the existence of solitary wave solutions is not known.

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 2575	2. GOVT ACCESSION NO. A134556	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) ON THE MODEL EQUATIONS WHICH DESCRIBE NONLINEAR WAVE MOTIONS IN A ROTATING FLUID	5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period	
	6. PERFORMING ORG. REPORT NUMBER	
7. AUTHOR(s) Jong Uhn Kim	8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041 and MCS-7927062, Mod. 2	
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Madison, Wisconsin 53706	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number - 1 - Applied Analysis	
11. CONTROLLING OFFICE NAME AND ADDRESS * see below	12. REPORT DATE September 1983	
	13. NUMBER OF PAGES 25	
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report) UNCLASSIFIED	
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE	
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES * U. S. Army Research Office National Science Foundation P. O. Box 12211 Washington, D.C. 20550 Research Triangle Park North Carolina 27709		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) nonlinear, dispersive waves in a rotating fluid; local existence of solution; formation of singularities; periodic travelling wave solutions		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) This paper is concerned about the mathematical aspects of the two model equations which describe nonlinear wave motions in a rotating fluid. We establish the local existence of solutions and show that singularities occur in a finite time under certain hypotheses. We also show that these equations admit nonconstant travelling wave solutions.		

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